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HYPEREXPANSIVE WEIGHTED TRANSLATION SEMIGROUPS

BY

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Abstract. Weighted shift operators turn out to be extremely useful in supplying interesting examples of operators on Hilbert spaces. With a view to studying continuous analogues of weighted shifts, M. Embry and A. Lambert initiated the study of operator semigroups $\{S_t\}$ indexed by non-negative real numbers and termed weighted translation semigroups, where the operators S_t are defined on $L^2(\mathbb{R}_+)$ by using a weight function. We obtain characterizations of hyperexpansive weighted translation semigroups in terms of their symbols. We also discuss the Cauchy dual of a hyperexpansive weighted translation semigroup. As an application, we present new proofs of a couple of known results.

1. Introduction. Let H be a complex separable Hilbert space and $\mathcal{B}(H)$ the algebra of bounded linear operators on H. Recall that for an orthonormal basis $\{e_n\}$ of H and a bounded sequence $\{\alpha_n\}$ of scalars, a weighted shift operator $T : \{\alpha_n\}$ is defined by $T(e_n) = \alpha_n e_{n+1}$ and extended by linearity and continuity. The flexibility of choosing $\{\alpha_n\}$ allows one to construct several interesting examples of a variety of classes of operators. The class of weighted shift operators has been systematically studied in [17]. With a view to developing a continuous analogue of weighted shifts, M. Embry and A. Lambert [11] initiated the study of operators that can be defined by using a *weight function* (rather than a weight sequence). In fact, for a positive, measurable function φ, they constructed a semigroup $\{S_t\}$ of bounded linear operators on $L^2(\mathbb{R}_+)$, parametrized by non-negative real numbers t and termed a *weighted translation semigroup.*

In this paper, we further explore the semigroups $\{S_t\}$, provide a variety of examples and present some important properties. In Section 2, we set the notation and record some definitions required in the sequel. In Section 3, we study some special types of weighted translation semigroups. In [11] and [12], M. Embry and A. Lambert characterized hyponormal and subnormal weighted translation semigroups. Here, capitalizing on the theory

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of associating a special type of non-negative function to a special type of bounded linear operator as developed in [5] and [20], we provide characterizations for completely hyperexpansive, 2-hyperexpansive, 2-isometric and alternatingly hyperexpansive weighted translation semigroups in terms of their symbols. In this paper, these four classes of operator semigroups are referred to as hyperexpansive weighted translation semigroups. In the process, we also obtain a characterization of a contractive subnormal weighted translation semigroup in terms of a completely monotone function. Section 4 is devoted to the notion of Cauchy dual of a hyperexpansive weighted translation semigroup. We present a comparative analysis of weighted shift operators and weighted translation semigroups, especially in the light of Cauchy duals of corresponding classes, and present new proofs of some known results.

2. Preliminaries. Let $\mathbb{R}_+$ be the set of non-negative real numbers and $L^2(\mathbb{R}_+)$ the Hilbert space of complex valued square integrable Lebesgue measurable functions on $\mathbb{R}_+$. Let $\mathcal{B}(L^2)$ denote the algebra of bounded linear operators on $L^2(\mathbb{R}_+)$.

Definition 2.1. For a measurable, positive function φ defined on $\mathbb{R}_+$ and $t \in \mathbb{R}_+$, define the function $\varphi_t : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\varphi_t(x) = \begin{cases} \sqrt{\dfrac{\varphi(x)}{\varphi(x-t)}} & \text{if } x \geq t, \\ 0 & \text{if } x < t. \end{cases}$$

Suppose that φ_t is essentially bounded for every $t \in \mathbb{R}_+$. For each fixed $t \in \mathbb{R}_+$, we define S_t on $L^2(\mathbb{R}_+)$ by

$$S_t f(x) = \begin{cases} \varphi_t(x) f(x-t) & \text{if } x \geq t, \\ 0 & \text{if } x < t. \end{cases}$$

Remark 2.2. It is easy to see that for every $t \in \mathbb{R}_+$, S_t is a bounded linear operator on $L^2(\mathbb{R}_+)$ with $\|S_t\| = \|\varphi_t\|_\infty$, where $\|\varphi_t\|_\infty$ stands for the essential supremum of φ_t. The family $\{S_t : t \in \mathbb{R}_+\}$ in $\mathcal{B}(L^2)$ is a semigroup with $S_0 = I$, the identity operator, and for all $t, s \in \mathbb{R}_+$, $S_t \circ S_s = S_{t+s}$.

We say that φ_t is the *weight function corresponding to the operator* S_t. Further, the semigroup $\{S_t : t \in \mathbb{R}_+\}$ is referred to as the *weighted translation semigroup with symbol* φ. Throughout this article, we assume that the symbol φ is a continuous function on $\mathbb{R}_+$. Further, wherever necessary, φ is assumed to be differentiable on $(0, \infty)$.

We now turn our attention towards extending the association of special classes of functions and special types of operator semigroups $\{S_t\}$. We recall the definitions of some special types of functions. A continuous function $f : \mathbb{R}_+ \to \mathbb{R}$ which is of class C^∞ on $(0, \infty)$ is called

(1) *completely monotone* if

$$(-1)^k f^{(k)}(x) \geq 0 \quad \text{for all } k \geq 0 \text{ and } x \in (0, \infty)$$

where $f^{(k)}$ denotes the kth derivative of f;

(2) *completely alternating* if

$$(-1)^{k-1} f^{(k)}(x) \geq 0 \quad \text{for all } k \geq 1 \text{ and } x \in (0, \infty);$$

(3) *absolutely monotone* if

$$f^{(k)}(x) \geq 0 \quad \text{for all } k \geq 0 \text{ and } x \in (0, \infty).$$

These classes of functions have been extensively studied in the literature [7], [16], [22] and they are naturally associated with some special classes of operators. Let T be a bounded linear operator on a Hilbert space H and n be a positive integer. Let $B_n(T)$ denote the operator

$$B_n(T) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} T^{*k} T^k. \tag{2.1}$$

An operator T is said to be

(1) *subnormal* if there exist a Hilbert space K containing H and a normal operator $N \in \mathcal{B}(K)$ such that $NH \subseteq H$ and $N|_H = T$;
(2) *completely hyperexpansive* if $B_n(T) \leq 0$ for all $n \geq 1$;
(3) *m-hyperexpansive* if $B_n(T) \leq 0$ for all $n \leq m$;
(4) *alternatingly hyperexpansive* if

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} T^{*k} T^k \geq 0 \quad \text{for all } n \geq 1;$$

(5) *m-isometric* if $B_m(T) = 0$;
(6) *hyponormal* if $T^*T - TT^* \geq 0$;
(7) a *contraction* [an *expansion*] if $I - T^*T \geq 0$ [$I - T^*T \leq 0$].

For a detailed account of these classes of operators, the reader is referred to [2], [5], [10], [14], [15], [20].

We now quote a characterization of subnormal contractions given by Agler [1].

THEOREM 2.3. *An operator T on a Hilbert space H is a subnormal contraction if and only if $B_n(T) \geq 0$ for all $n \geq 1$.*

Among the classes of operators defined above, hyponormal and subnormal weighted translation semigroups have been studied by M. Embry and A. Lambert. Their characterizations in terms of symbols are recalled below ([11, Lemma 3.3], [12, Theorem 2.2]).

THEOREM 2.4. *The semigroup $\{S_t\}$ with a continuous, positive symbol φ is hyponormal if and only if $\log \varphi$ is convex.*

The assumption of continuity of φ in the above theorem is superfluous. It turns out that if the semigroup $\{S_t\}$ is hyponormal, then the function $\log\varphi$ is mid-point convex, and by a result of Sierpiński [21], a measurable and mid-point convex function is continuous.

THEOREM 2.5. *A semigroup $\{S_t\}$ with a continuous symbol φ is subnormal if and only if*

$$\varphi(x) = \int_0^a s^x \, d\rho(s),$$

where $a = \lim_{t\to\infty} \|S_t^* S_t\|^{1/t}$ and ρ is a probability measure on $[0, a]$.

3. Characterizations of hyperexpansive weighted translation semigroups. In this section, we continue the theme of characterizing some classes of weighted translation semigroups in terms of their symbols. We say that a semigroup $\{S_t\}$ is *2-hyperexpansive*, *m-isometric*, *completely hyperexpansive*, *subnormal*, *hyponormal*, *alternatingly hyperexpansive* if each operator S_t has the respective property. Consider a weighted translation semigroup $\{S_t\}$ with symbol φ. Observe that the adjoint of S_t is given by

$$S_t^* f(x) = \sqrt{\frac{\varphi(x+t)}{\varphi(x)}} f(x+t) \quad \text{for almost every } x \in \mathbb{R}_+.$$

Further it is easy to see that

$$(S_t^* S_t) f(x) = \frac{\varphi(x+t)}{\varphi(x)} f(x) \quad \text{for almost every } x \in \mathbb{R}_+.$$

THEOREM 3.1. *Let $\{S_t\}$ be a weighted translation semigroup with symbol $\varphi \in C^n$ for a fixed positive integer n. Then the following statements are equivalent:*

(1) $B_n(S_t) \geq 0$ *for all* $t \in \mathbb{R}_+$.
(2) $\sum_{k=0}^{n} (-1)^k \binom{n}{k} \varphi(x+kt) \geq 0$ *for all* $x, t \in \mathbb{R}_+$.
(3) $(-1)^n \varphi^{(n)}(x) \geq 0$ *for all* $x \in (0, \infty)$.

Proof. We first prove (1)$\Leftrightarrow$(2). Note that for each $t \in \mathbb{R}_+,\ B_n(S_t) \geq 0$ if and only if $\langle B_n(S_t) f, f\rangle \geq 0$ for all $f \in L^2(\mathbb{R}_+)$. This is true if and only if

$$\int_0^\infty \Big(\sum_{k=0}^{n} (-1)^k \binom{n}{k} \varphi(x+kt) \Big) |f(x)|^2 \, dx \geq 0 \quad \text{for all } f \in L^2(\mathbb{R}_+),\ x, t \in \mathbb{R}_+.$$

Now by continuity of φ, this is equivalent to (2).

We now prove (2)$\Rightarrow$(3). Since $\varphi \in C^n$, we have

$$(-1)^n \varphi^{(n)}(x) = \lim_{h\to 0} \frac{\sum_{k=0}^{n} (-1)^k \binom{n}{k} \varphi(x+kh)}{h^n}.$$

Hence the condition

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\varphi(x+kt)\geq 0 \quad \text{for all } x,t\in\mathbb{R}_+$$

implies that $(-1)^n\varphi^{(n)}(x)\geq 0$ for all $x\in(0,\infty)$.

(3)⇒(2) follows from the fact that repeated application of the Mean Value Theorem gives

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\varphi(x+kt)=(-1)^n t^n\varphi^{(n)}(x')$$

for some $x'\in(x,x+nt)$ [16, Theorem 4.8]. ■

The proof of the following theorem is similar to the above.

THEOREM 3.2. *Let $\{S_t\}$ be a weighted translation semigroup with symbol $\varphi\in C^n$ for a fixed positive integer n. Then the following statements are equivalent:*

(1) $B_n(S_t)\leq 0$ *for all* $t\in\mathbb{R}_+$.
(2) $\sum_{k=0}^{n}(-1)^k\binom{n}{k}\varphi(x+kt)\leq 0$ *for all* $x,t\in\mathbb{R}_+$.
(3) $(-1)^n\varphi^{(n)}(x)\leq 0$ *for all* $x\in(0,\infty)$.

The following corollary follows at once from Theorems 3.1 and 3.2.

COROLLARY 3.3. *Let $\{S_t\}$ be a weighted translation semigroup with symbol $\varphi\in C^\infty$. Then the semigroup $\{S_t\}$ is*

(1) *2-hyperexpansive if and only if φ is concave;*
(2) *m-isometric if and only if φ is a polynomial of degree $m-1$;*
(3) *completely hyperexpansive if and only if φ is completely alternating;*
(4) *a subnormal contraction if and only if φ is completely monotone;*
(5) *alternatingly hyperexpansive if and only if φ is absolutely monotone.*

REMARK 3.4. Note that if the semigroup $\{S_t\}$ is 2-hyperexpansive, then the symbol φ is mid-point concave. As in the case of convexity, it turns out that a measurable, mid-point concave function is continuous and hence concave. Thus statement (1) in Corollary 3.3 can be proved only under the assumption of measurability of φ. Also (2) can be proved under the weaker condition that $\varphi\in C^m$.

We now give characterizations of contractive subnormal and completely hyperexpansive semigroups $\{S_t\}$, in terms of integral representations of the symbol φ, by appealing to a suitable modification of the integral representations of completely monotone and completely alternating maps as given in [16, Theorems 1.4, 3.2] for functions defined and continuous on $\mathbb{R}_+$, and C^∞ on $(0,\infty)$.

Proposition 3.5. *Let $\{S_t\}$ be a weighted translation semigroup with symbol $\varphi \in C^\infty$.*

(1) *The semigroup $\{S_t\}$ is a subnormal contraction if and only if*

$$\varphi(x) = \int_0^\infty e^{-xa}\, d\mu(a), \qquad x \in (0, \infty),$$

where μ is a finite non-negative Borel measure on $[0, \infty)$.

(2) *The semigroup $\{S_t\}$ is completely hyperexpansive if and only if*

$$\varphi(x) = \varphi(0) + cx + \int_0^\infty (1 - e^{-ax})\, d\mu(a), \qquad x \in (0, \infty),$$

where c is a non-negative real number and μ is a measure on $[0, \infty)$ satisfying $\int_0^\infty (1 \wedge a)\, d\mu(a) < \infty$. Here, $1 \wedge a = \min\{1, a\}$.

Remark 3.6. We now show that if φ is either completely monotone or completely alternating, then the weight function φ_t is essentially bounded.

(1) If φ is completely monotone, then $\varphi' \le 0$. Therefore φ is decreasing. Hence $\varphi_t \le 1$ for all $t \in \mathbb{R}_+$.

(2) If φ is completely alternating, then $\varphi' \ge 0$ and therefore φ is increasing. Hence $\varphi_t(x) \ge 1$ for $x \ge t$. We now prove that for every fixed $t \in \mathbb{R}_+, \lim_{x\to\infty} \varphi_t(x) = 1$. Since $\varphi_t \ge 0$, it is sufficient to prove that

$$\lim_{x\to\infty} \varphi_t^2(x) = \lim_{x\to\infty} \frac{\varphi(x)}{\varphi(x-t)} = 1.$$

Since φ is completely alternating, by Corollary 3.3(3) and Proposition 3.5(2) we have

$$\varphi(x) = \varphi(0) + cx + \int_0^\infty (1 - e^{-ax})\, d\mu(a),$$

where $c \ge 0$ and μ is a measure on $[0, \infty)$ with $\int_0^\infty (1 \wedge a)\, d\mu(a) < \infty$. Now,

$$\begin{aligned}\frac{\varphi(x)}{\varphi(x-t)} - 1 &= \frac{\varphi(x) - \varphi(x-t)}{\varphi(x-t)}\\ &= \frac{ct + \int_0^\infty [e^{-a(x-t)} - e^{-ax}]\, d\mu(a)}{\varphi(0) + c(x-t) + \int_0^\infty (1 - e^{-a(x-t)})\, d\mu(a)}.\end{aligned}$$

Note that $F(x) = e^{-a(x-t)} - e^{-ax}$ is a decreasing function and $F(x) \to 0$ as $x \to \infty$. By the Lebesgue monotone convergence theorem, it follows that $\int_0^\infty [e^{-a(x-t)} - e^{-ax}]\, d\mu(a) \to 0$. Hence $\lim_{x\to\infty} \varphi_t(x) = 1$.

As noted in Corollary 3.3(1), the symbol φ of a 2-hyperexpansive weighted translation semigroup is concave. By [9, Lemma 3.16], if φ is C^2, then it is increasing. Recall that if a weighted shift operator is 2-hyperexpansive, then the corresponding weight sequence is decreasing [5, Proposition 4]. However, the following example indicates that the weight function φ_t associated to a 2-hyperexpansive operator S_t need not be monotone.

EXAMPLE 3.7. Let $\varphi(x) = \sqrt{x+1}$. As φ is concave, by Corollary 3.3(1) the corresponding semigroup $\{S_t\}$ is 2-hyperexpansive. A simple computation reveals that φ_1, the weight function corresponding to S_1, is neither increasing nor decreasing.

By choosing special types of symbols, it is easy to construct special types of weighted translation semigroups. Here, we illustrate Corollary 3.3 in some special cases.

EXAMPLE 3.8. Let φ be the symbol of the semigroup $\{S_t\}$. In the following examples, the weight function φ_t is essentially bounded for every $t \in \mathbb{R}_+$.

(1) Let $\varphi(x) = \frac{x+\lambda}{x+1}$, $\lambda > 0$. Then $\varphi^{(n)}(x) = \frac{(1-\lambda)(-1)^{n-1}n!}{(x+1)^{n+1}}$. For $0 < \lambda < 1$, the symbol φ is completely alternating and the corresponding semigroup $\{S_t\}$ is completely hyperexpansive. For $\lambda > 1$, the symbol φ is completely monotone and $\{S_t\}$ is a subnormal contraction.
(2) Let $\varphi(x) = e^x$. Since $\log\varphi$ is convex, $\{S_t\}$ is hyponormal. By Corollary 3.3(4), this semigroup is not subnormal.

In [13, Theorem 1], it was proved that if a function φ is completely monotone, then $\log\varphi$ is convex. We now present a different proof of this fact using Corollary 3.3.

PROPOSITION 3.9. *If a function φ is completely monotone, then* $\log\varphi$ *is convex.*

Proof. Suppose a positive C^∞ function φ is completely monotone. Then the weight function φ_t is essentially bounded. Therefore by Corollary 3.3(4), the semigroup $\{S_t\}$ with symbol φ is subnormal. This implies that $\{S_t\}$ is hyponormal [10, Proposition 4.2]. Hence by Theorem 2.4, the function $\log\varphi$ is convex. ∎

4. Cauchy dual of hyperexpansive weighted translation semigroups. The notion of Cauchy dual of a left invertible operator was introduced by S. Shimorin [18]. Recall that for a left invertible operator T, the *Cauchy dual* T' of T is defined as $T' = T(T^*T)^{-1}$. Note that an operator T is left invertible if and only if T is injective and its range is closed.

In this section, we shall discuss the Cauchy dual of a hyperexpansive weighted translation semigroup. In particular, the Cauchy duals of completely hyperexpansive, 2-hyperexpansive and 2-isometric weighted translation semigroups will be discussed. Though the Cauchy duals of each of these classes of operators have been dealt with in the literature [3], [5],

[8], [19], function-theoretic considerations allow one to give considerably simpler proofs in the context of a weighted translation semigroup.

Let $\{S_t\}$ be a weighted translation semigroup with symbol φ. It is easy to check that for every $t \in \mathbb{R}_+$, the operator S_t is injective. Further, if we assume that for every $t \in \mathbb{R}_+$, $\inf_x \frac{\varphi(x+t)}{\varphi(x)} > 0$, then the range of S_t is closed, implying that S_t is left invertible for every $t \in \mathbb{R}_+$. We say that the semigroup $\{S_t\}$ is *left invertible* if each operator S_t is left invertible. Thus the stated condition on φ ensures the left invertibility of the semigroup $\{S_t\}$. In this case, the operator $(S_t^* S_t)^{-1} S_t^*$ is a left inverse of S_t. Observe that for any $t \in \mathbb{R}_+$,

$$(S_t^* S_t)^{-1} f(x) = \frac{\varphi(x)}{\varphi(x+t)} f(x) \quad \text{for almost every } x \in \mathbb{R}_+.$$

Now it is easy to see that the Cauchy dual S_t' of S_t is given by

$$S_t' f(x) = \begin{cases} \frac{1}{\varphi_t(x)} f(x-t) & \text{if } \ x \geq t, \\ 0 & \text{if } \ x < t. \end{cases}$$

Observe that for $t \in \mathbb{R}_+$, the family $\{S_t'\}$ of operators also forms a semigroup. We say that the weighted translation semigroup $\{S_t'\}$ is the *Cauchy dual* of the weighted translation semigroup $\{S_t\}$.

REMARK 4.1. Note that if $\{S_t\}$ is a weighted translation semigroup with symbol φ, then the symbol corresponding to the weighted translation semigroup $\{S_t'\}$ is $1/\varphi$. Observe that if the weight function of a left invertible operator S_t is φ_t, then the weight function of its Cauchy dual S_t' is $1/\varphi_t$. This resembles the fact that the weight sequence of the Cauchy dual of a left invertible weighted shift $T : \{\alpha_n\}$ is $\{1/\alpha_n\}$.

REMARK 4.2. If $\{S_t\}$ is a hyperexpansive weighted translation semigroup with symbol φ, then φ is increasing and $\inf_x \frac{\varphi(x+t)}{\varphi(x)} \geq 1$ for all $t \in \mathbb{R}_+$. Thus every hyperexpansive weighted translation semigroup is left invertible.

It is known that if T is a completely hyperexpansive weighted shift, then its Cauchy dual T' is a subnormal contraction [5, Remark 4, Proposition 6]. We now prove this result in the context of a weighted translation semigroup.

PROPOSITION 4.3. *If the semigroup $\{S_t\}$ with symbol $\varphi \in C^\infty$ is a completely hyperexpansive weighted translation semigroup, then its Cauchy dual $\{S_t'\}$ is a subnormal contraction.*

Proof. In view of Corollary 3.3(3, 4) and Remark 4.1, it is sufficient to prove that if φ is completely alternating, then $1/\varphi$ is completely monotone. This result is a special case of [16, Theorem 3.6(ii)]. However, we give a direct and simple proof. Let $\psi = 1/\varphi$. Then

$$\psi' = \frac{-1}{\varphi^2}\varphi' \le 0, \qquad \psi'' = \frac{-1}{\varphi^2}\varphi'' + \frac{2}{\varphi^3}\varphi'^2 \ge 0.$$

Suppose k and n are positive integers. Suppose $\psi^{(k)} \le 0$ if k is odd and $\psi^{(k)} \ge 0$ if k is even for $k < n$. We now want to prove that $\psi^{(n)} \le 0$ if n is odd and $\psi^{(n)} \ge 0$ if n is even. Note that $\psi\varphi = 1$. Therefore $(\psi\varphi)^{(n)} = 0$. Thus

$$\psi^{(n)}\varphi = -\binom{n}{1}\psi^{(n-1)}\varphi' - \binom{n}{2}\psi^{(n-2)}\varphi'' - \cdots - \binom{n}{n-1}\psi'\varphi^{(n-1)} - \psi\varphi^{(n)}.$$

Each term on the right hand side is non-positive if n is odd, and each is non-negative if n is even. Therefore $\psi^{(n)} \le 0$ if n is odd and $\psi^{(n)} \ge 0$ if n is even. Hence $\psi = 1/\varphi$ is completely monotone. ∎

Remark 4.4. The technique used in the proof of Proposition 4.3 allows one to give a simpler proof of the fact that if T is a completely hyperexpansive weighted shift, then its Cauchy dual T' is a subnormal contraction. The sequence $\{\beta_n\}$ associated with the weighted shift operator $T : \{\alpha_n\}$ is defined by $\beta_0 = 1, \beta_n = \prod_{k=0}^{n-1} \alpha_k^2 \quad (n \ge 1)$. Note that $\beta_n = \|T^n e_0\|^2$ and $\alpha_n = \sqrt{\beta_{n+1}/\beta_n}$, $n \ge 0$. Recall that a weighted shift T is completely hyperexpansive [a subnormal contraction] if and only if the sequence $\{\beta_n\}$ is completely alternating [completely monotone]. The forward difference operator on $\mathbb{N}$ is defined as $(\Delta\phi)(n) = \phi(n+1) - \phi(n)$. The map ϕ is completely alternating if and only if $(-1)^k\Delta^k\phi(n) \le 0$, and ϕ is completely monotone if and only if $(-1)^k\Delta^k\phi(n) \ge 0$. Let $\varphi(n) = \beta_n$ and $\psi(n) = 1/\varphi(n)$. The formula

$$(\Delta^n\phi\psi)(x) = \sum_{k=0}^{n}\binom{n}{k}(\Delta^k\phi)(x)(\Delta^{n-k}\psi)(x+k)$$

allows one to use the argument similar to that in the proof of Proposition 4.3 to show that if the sequence $\varphi(n)$ is completely alternating, then the sequence $\psi(n) = 1/\varphi(n)$ is completely monotone.

Remark 4.5. In view of Corollary 3.3 and Proposition 4.3, it is instructive to give a procedure for generating a weighted shift operator from a weighted translation semigroup and vice versa in the context of a completely hyperexpansive weighted translation semigroup and its Cauchy dual. Let $\{S_t\}$ be a completely hyperexpansive weighted translation semigroup with symbol $\varphi \in C^\infty$. Define

$$\beta_n = \varphi(n) \quad \text{and} \quad \alpha_n = \sqrt{\frac{\beta_{n+1}}{\beta_n}} = \sqrt{\frac{\varphi(n+1)}{\varphi(n)}}.$$

It is clear that the weighted shift operator with weight sequence $\{\alpha_n\}$ is completely hyperexpansive. For the reverse process, we need to start with a completely hyperexpansive weighted shift operator T such that the corresponding completely alternating sequence $\{\beta_n\}_{n\geq 0}$ is minimal. Recall that a sequence $\{\beta_n\}$ is said to be *minimal* if the sequence $\{\beta_0, \beta_1-\epsilon, \beta_2-\epsilon, \ldots\}$ is not completely alternating for any positive ϵ. By an application of [6, Theorem 1], there exists a completely alternating function φ on $\mathbb{R}_+$ satisfying $\varphi(n) = \beta_n$. Now the semigroup $\{S_t\}$ with symbol φ is clearly a completely hyperexpansive weighted translation semigroup. One may apply a similar procedure with complete hyperexpansion replaced by a subnormal contraction with appropriate changes in the definition of minimality of a completely monotone sequence. The details can be carried out by using [22, Chapter 4, Definition 14a, Theorem 14b]. The following diagram depicts the association of completely hyperexpansive weighted shift operators and completely hyperexpansive weighted translation semigroups as well as their respective Cauchy duals, as described above:

$$\begin{array}{ccccccc} T:\{\alpha_n\} & \longleftrightarrow & \{\beta_n\} & \longleftrightarrow & \varphi(x) & \longleftrightarrow & \{S_t\} \\ \updownarrow & & & & & & \updownarrow \\ T':\left\{\frac{1}{\alpha_n}\right\} & \longleftrightarrow & \left\{\frac{1}{\beta_n}\right\} & \longleftrightarrow & \frac{1}{\varphi(x)} & \longleftrightarrow & \{S'_t\} \end{array}$$

In the light of the positive result in the case of a weighted shift operator, one might expect that the Cauchy dual of a completely hyperexpansive operator is a subnormal contraction. However, an example of a 2-isometry whose Cauchy dual is not subnormal has recently been constructed in [4]. It was proved that the Cauchy dual of a 2-isometric weighted shift is a subnormal contraction [3, Theorem 2.5(3)]. The proof of this fact in the special case of a 2-isometric weighted translation semigroup is trivial.

We now turn our attention to the class 2-hyperexpansive weighted translation semigroups. Note that the class of 2-hyperexpansive operators is strictly bigger than the class of completely hyperexpansive operators. In [19] and [8, Theorem 2.9], it is proved that if T is a 2-hyperexpansive operator, then its Cauchy dual is a hyponormal contraction. We now present a special case of this result for a 2-hyperexpansive weighted translation semigroup.

PROPOSITION 4.6. *If the semigroup $\{S_t\}$ with symbol $\varphi \in C^2$ is a 2-hyperexpansive weighted translation semigroup, then its Cauchy dual $\{S'_t\}$ is a hyponormal contraction.*

Proof. In view of Corollary 3.3(1), Theorem 2.4 and Remark 4.1, it is sufficient to prove that if φ is a concave function, then $\log\frac{1}{\varphi}$ is a convex function. The proof is straightforward. ∎

REMARK 4.7. If $\{S_t\}$ is a weighted translation semigroup, then the semigroup property implies that for any non-negative integer n, the operator $S_t^n = S_{nt}$ again belongs to the semigroup $\{S_t\}$. In particular, if $\{S_t\}$ is a hyponormal semigroup, then each operator S_t is power hyponormal. Thus the Cauchy dual S_t' of a 2-hyperexpansive operator S_t is power hyponormal.

Note that the Cauchy dual of a completely hyperexpansive weighted translation semigroup is a subnormal contraction (Proposition 4.3) and the Cauchy dual of a 2-hyperexpansive weighted translation semigroup is a hyponormal contraction (Proposition 4.6). In this context, we now present an example of a 2-hyperexpansive weighted translation semigroup $\{S_t\}$ whose Cauchy dual is not a subnormal contraction. In view of Remark 4.7, the Cauchy dual operator S_t' in the following example is power hyponormal which is not subnormal.

EXAMPLE 4.8. Let $\{S_t\}$ be a weighted translation semigroup with symbol φ given by $\varphi(x) = 2x - \log(\cosh(x-10)) + 100$. Observe that $\varphi > 0$, $\varphi \in C^\infty$ and $\varphi'(x) = 2 - \tanh(x-10) \geq 0$, $\varphi''(x) = \tanh^2(x-10) - 1 \leq 0$. Therefore φ is concave, implying that the semigroup $\{S_t\}$ is 2-hyperexpansive. Now $\varphi'''(x) = 2\tanh(x-10)(1-\tanh^2(x-10))$. Observe that $\varphi'''(11) > 0$ and $\varphi'''(0.001) < 0$. Hence φ is not completely alternating. Consequently, $\{S_t\}$ is not completely hyperexpansive. Let $\psi(x) = 1/\varphi(x)$. Then $\psi'(x) \leq 0$ and $\psi''(x) \geq 0$. By direct computations, we get $\psi'''(11) < 0$ and $\psi'''(0.001) > 0$. Hence ψ is not completely monotone, and thus $\{S_t'\}$ is not a subnormal contraction.

The problem of describing the Cauchy dual of an alternatingly hyperexpansive operator, even in the special case of a weighted translation semigroup, seems to be difficult. The following examples illustrate the situation in some simple cases. In view of Corollary 3.3(5) and Remark 4.1, we need to examine the reciprocal of an absolutely monotone function.

EXAMPLE 4.9.

(1) Let $\varphi(x) = e^x$. It is an absolutely monotone function. Observe that the function $1/\varphi(x) = e^{-x}$ is completely monotone.
(2) The following result is a special case of [3, Proposition 4.3]. Let $\varphi(x) = x^2 + px + q,\ p, q > 0$. It is an absolutely monotone function. Suppose $\varphi(x)$ has both roots complex. It is easy to see that the function $1/\varphi$ is not completely monotone.
(3) Let $\varphi(x) = x^2 + px + q,\ p, q > 0$. Suppose $\varphi(x)$ has both roots real. It can be seen that the function $1/\varphi$ is completely monotone.

5. Concluding remarks. The present work attempts to study the behaviour of an operator S_t with a motivation to compare it with a weighted shift operator. The appearance of a function in place of a sequence allows one to use different techniques, not available in the discrete case. The authors believe that these techniques might be employed to tackle some problems about weighted shifts. In particular, the problem of characterizing m-isomeric transformations whose Cauchy dual is a contractive subnormal operator is challenging even in the case of a weighted shift operator. Though some special cases of this problem have been discussed in [3], the general case still remains unanswered. In a sequel to this article, the authors have developed an analytic model for left invertible weighted translation semigroups enabling one to realize every left invertible operator S_t as multiplication by z on some suitable reproducing kernel Hilbert space. In particular, the model applies to hyperexpansive weighted translation semigroups. We also describe the spectral picture of a left invertible weighted translation semigroup. In the process, we point out that in contrast with the situation for weighted shift operators, the kernel of the adjoint S_t^* is infinite-dimensional.

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REFERENCES

[1] J. Agler, *Hypercontractions and subnormality*, J. Operator Theory 13 (1985), 203–217.
[2] J. Agler and M. Stankus, *m-isometric transformations of Hilbert spaces, I–III*, Integral Equations Operator Theory 21, 23, 24 (1995, 1995, 1996), 383–429, 1–48, 379–421.
[3] A. Anand and S. Chavan, *A moment problem and joint q-isometry tuples*, Complex Anal. Operator Theory 11 (2017), 785–810.
[4] A. Anand, S. Chavan, Z. Jabłoński and J. Stochel, *A solution to the Cauchy dual subnormality problem for 2-isometries*, arXiv:1702.01264v4 (2018).
[5] A. Athavale, *On completely hyperexpansive operators*, Proc. Amer. Math. Soc. 124 (1996), 3745–3752.
[6] A. Athavale and A. Ranjekar, *Berstein functions, complete hyperexpansivity and subnormality - I*, Integral Equations Operator Theory 43 (2002), 253–263.
[7] C. Berg, J. P. R. Christensen and P. Ressel, *Harmonic Analysis on Semigroups*, Springer, Berlin, 1984.
[8] S. Chavan, *On operators Cauchy dual to 2-hyperexpansive operators*, Proc. Edinburgh Math. Soc. 50 (2007), 637–652.
[9] S. Chavan and V. Sholapurkar, *Completely monotone functions of finite order and Agler's conditions*, Studia Math. 226 (2015), 229–258.
[10] J. Conway, *The Theory of Subnormal Operators*, Math. Surveys Monogr. 36, Amer. Math. Soc., Providence, RI, 1991.
[11] M. Embry and A. Lambert, *Weighted translation semigroups*, Rocky Mountain J. Math. 7 (1977), 333–344.
[12] M. Embry and A. Lambert, *Subnormal weighted translation semigroups*, J. Funct. Anal. 24 (1977), 268–275.

[13] A. M. Fink, *Kolmogorov–Landau inequalities for monotone functions*, J. Math. Anal. Appl. 90 (1982), 251–258.
[14] Z. Jabłoński, *Complete hyperexpansivity, subnormality and inverted boundedness conditions*, Integral Equations Operator Theory 44 (2002), 316–336.
[15] Z. Jabłoński, *Hyperexpansive operator-valued unilateral weighted shifts*, Glasgow Math. J. 46 (2004), 405–416.
[16] R. L. Schilling, R. Song and Z. Vondraček, *Bernstein Functions. Theory and Applications*, de Gruyter Stud. Math. 37, de Gruyter, 2012.
[17] A. Shields, *Weighted shift operators and analytic function theory*, in: Topics in Operator Theory, Math. Surveys 13, Amer. Math. Soc., Providence, RI, 1974, 49–128.
[18] S. Shimorin, *Wold-type decompositions and wandering subspaces for operators close to isometries*, J. Reine Angew. Math. 531 (2001), 147–189.
[19] S. Shimorin, *Complete Nevanlinna–Pick property of Dirichlet-type spaces*, J. Funct. Anal. 191 (2002), 276–296.
[20] V. Sholapurkar and A. Athavale, *Completely and alternatingly hyperexpansive operators*, J. Operator Theory 43 (2000), 43–68.
[21] W. Sierpiński, *Sur les fonctions convexes mesurables*, Fund. Math. 1 (1920), 125–129.
[22] D. Widder, *The Laplace Transform*, Princeton Univ. Press, London, 1946.

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